CRM Montreal 2022

Quantum Walks and Graph Coloring

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Monday

Outline of the mini-course

- **Review on graph coloring**
- Coined quantum walks on graphs
- Staggered quantum walks on graphs
- Spatial search algorithms
- Equivalence of discrete-time QWs

Today's Outline

- Review on graph coloring
  - Basic definitions
  - Clique and clique graph
  - Line graph
  - Vertex coloring
  - Edge coloring
  - Basic definitions on digraphs

Motivation

The main motivation for discussing graph coloring:

- The definition of Coined Quantum Walks and Staggered Quantum Walks requires many kind of labels
- Graph coloring is the art of giving labels to graph parts

On the other hand:

- The definition of Szegedy's Quantum Walk depends on the syntax of Markov chains on bipartite graphs
- The study of Continuous-time Quantum Walk depends on spectral graph theory (almost no need of graph coloring)

Basic graph theory [1,2,3]

We assume that some basic definitions are known: graph $G(V, E)$, vertex set V, edge set E, endpoints, neighborhood $N(v)$, adjacency, loop, multiple edges (multigraph), vertex degree $d(v)$, maximum degree $\Delta(G)$, d-regular graph, directed edges (digraph).

Definition

A *simple graph* is a loopless graph with no multiple edges.

Basic definitions: path, cycle, path length, connected graph, subgraph.

Definition

Let $V' \subset V$ and $E' \subset E$. An *induced subgraph* $H(V', E')$ is a subgraph of $G(V, E)$ if H has exactly the edges that appear in G over the vertex set V'.

[1] D. West. *Introduction to Graph Theory*.
[2] J.A. Bondy, U.S.R. Murty. *Graph Theory*
[3] Key papers

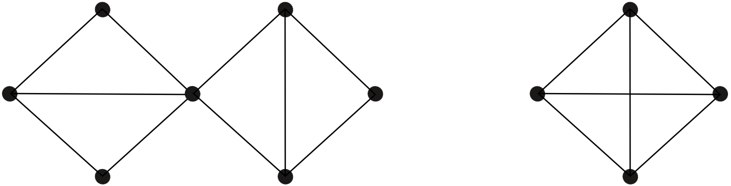
Basic graph theory

Definition

G is *H-free* if *G* has no induced subgraph *isomorphic* to graph *H*.

Example:

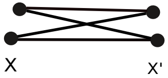
1. *G* (a graph with two diamonds) is not diamond-free.
2. K_4 is diamond-free.



G $\qquad$ K_4

Basic graph theory

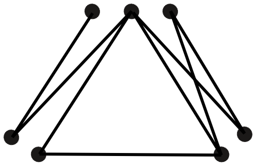
A *bipartite graph* is a graph whose vertex set V is the union of two disjoint sets X and X' so that no two vertices in X are adjacent and no two vertices in X' are adjacent.



Theorem

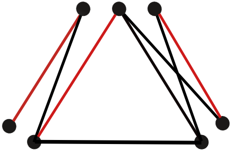
(König) G is bipartite if and only if G has no odd cycles.

An *independent set* or *stable set* is a set of pairwise nonadjacent vertices.



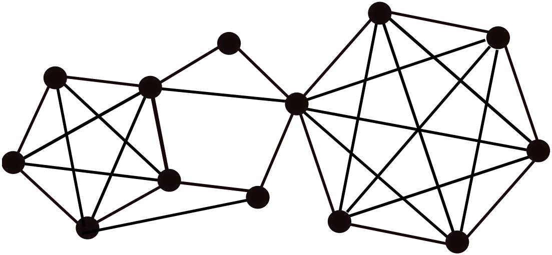
Basic graph theory

A *matching* $M \subseteq E$ is a set of edges without pairwise common vertices.
A *perfect matching* is a matching that matches all vertices of the graph.



Basic graph theory

A *clique* is a subset of vertices of a graph such that its induced subgraph is complete. A *maximal clique* is not contained in a larger clique.



A *partition* of V into cliques (*clique-partitioning*) is a set of cliques of G such that each vertex is represented in exactly one clique.

A set is independent if and only if it is a clique in the graph's complement.

Graph Operations

A (unary) graph operation

$$\mathcal{O} : \mathcal{C} \longrightarrow \mathcal{C}$$

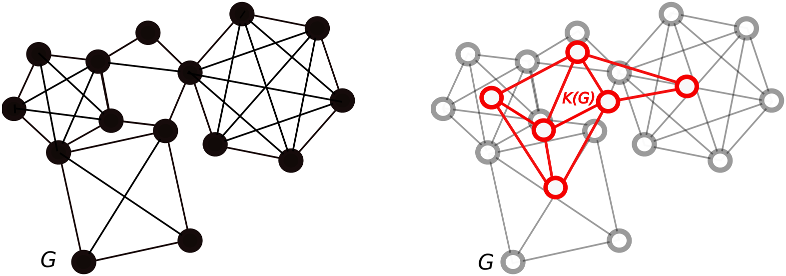
is a function on $\mathcal{C}$ (set of all graphs). $\mathcal{O}$ need not be injective.

Example: *Clique Graph Operation*

A *clique graph* $K(G)$ is a graph such that every vertex represents a maximal clique of G and two vertices of $K(G)$ are adjacent if and only if the underlying maximal cliques in G share at least one vertex in common.

Example of a clique graph

Clique graph $K(G)$ of graph G:



Each maximal clique becomes a red vertex.

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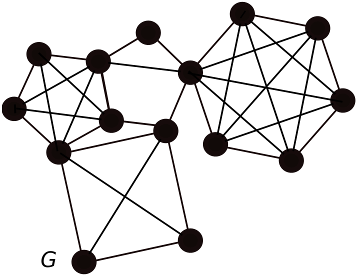
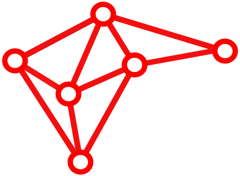




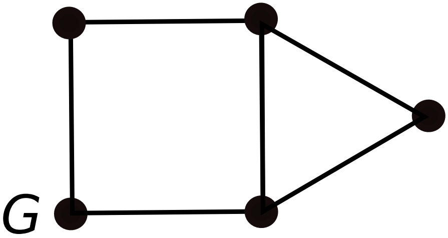
Each maximal clique becomes a red vertex.

Line Graph Operation

Definition

A *line graph* (or *derived graph* or *interchange graph*) of a graph G (called *root graph*) is another graph $L(G)$ so that each vertex of $L(G)$ represents an edge of G and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common vertex in G.

Example:



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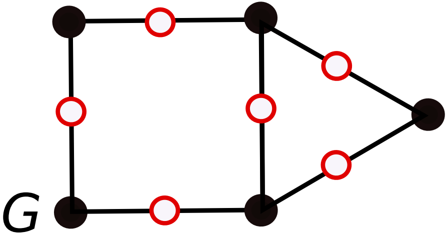

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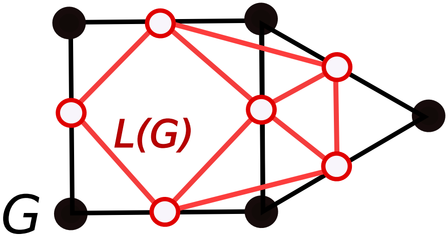


Line Graph Operation

Proposition

The clique graph of a connected *triangle-free graph G* is isomorphic to the line graph of G.

Proposition

The line graph of a multigraph is a simple graph.

Determining whether G is a line graph

Theorem

(Beineke) There exists H such that $G = L(H)$ if and only if G contains no forbidden Beineke graph as an induced subgraph.

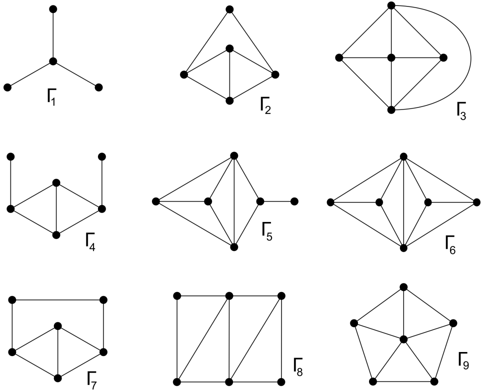


Figure: Nine forbidden Beineke graphs

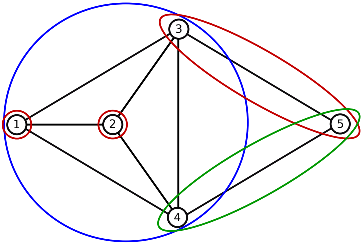
Determining whether G is a line graph

Definition

A Krausz partition is a collection C of subgraphs of G that satisfies:

1. Each element of C is a clique,
2. Each edge of G is in exactly one element of C,
3. Each vertex is in exactly two elements of C.

Example:



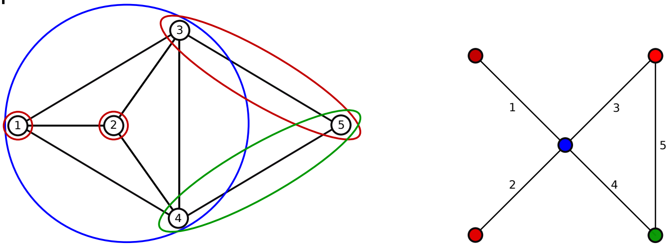
$$C = \{\{1, 2, 3, 4\}, \{3, 5\}, \{4, 5\}, \{1\}, \{2\}\}$$

Determining whether G is a line graph

Theorem

(Krausz) A graph G is a line graph of some graph H if and only if G has a Krausz partition.

Example:



Determining whether G is a line graph of a bipartite

Proposition

The line graph of H has a 2-colorable Krausz partition if and only if H is bipartite.

Proof.

Use the following results by Peterson. □

Theorem

(Peterson) G is the line graph of a bipartite graph if and only if G is diamond-free and $K(G)$ is bipartite.

Lemma

(Peterson) A graph is diamond-free if and only if any two cliques intersect in at most one vertex, which holds if and only if each edge lies in exactly one clique.

[1] D. Peterson. Gridline graphs. Discrete Applied Mathematics 126 (2003) 223–239.

Line Graph of Bipartite Multigraphs

It is possible to determine whether G is the line graph of a *bipartite multigraph* via the following theorem.

Theorem

(Peterson) A simple graph G is a line graph of a bipartite multigraph if and only if $K(G)$ is bipartite.

Graph coloring: vertex-coloring and edge-coloring

Definition

1. A (proper) *vertex-coloring* is a labeling of the vertices with colors so that no two vertices sharing the same edge have the same color.
2. The *chromatic number* $\chi(G)$ is the least number of colors needed to color a graph G.
3. A graph that can be assigned a coloring with k colors is *k-colorable*.
4. It is *k-chromatic* if its chromatic number is exactly k.

Theorem

(Brooks) $\chi(G) \leq \Delta(G)$ for a graph G, unless G is a complete graph or an odd cycle.

Edge-Coloring of a loopless graph

Definition

1. A (proper) *k-edge-coloring* is a coloring of the edges with k colors so that two edges in the same color class do not share a common vertex.
2. A graph is *k-edge-colorable* if it has an edge-coloring with k colors.
3. The *edge-chromatic number* or *chromatic index* $\chi'(G)$ is the least number k such that G is k-edge colorable.

Fact

1. *It is straightforward to check that* $\chi'(G) \geq \Delta(G)$.
2. *On the other hand,* $\Delta(G) + 1$ *is an upper bound for* $\chi'(G)$.
3. *An edge coloring of* G *is equivalent to a vertex coloring of* $L(G)$.

Edge-Coloring

Theorem

(Vizing) The edge-chromatic number of a simple graph G is either $\Delta(G)$ or $\Delta(G) + 1$, that is, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

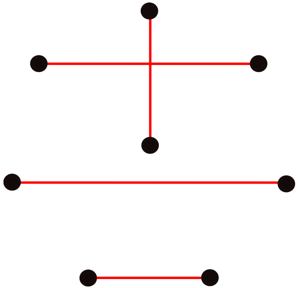
Classes:

- *Class* 1 graphs for which $\Delta(G)$ colors are sufficient
- *Class* 2 graphs for which $\Delta(G) + 1$ colors are necessary

Examples:

- Class 1: Complete graphs K_N for even N, bipartite graphs, bridgeless planar cubic graph.
- Class 2: Regular graphs with an odd number of vertices $N > 1$ (includes complete graphs K_N for odd $N \geq 3$), Petersen graph, snarks (bridgeless cubic and of class 2).

Example of class-1: K_N for even N



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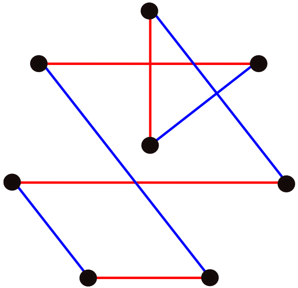

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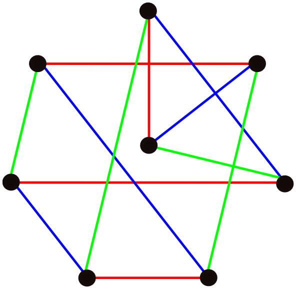

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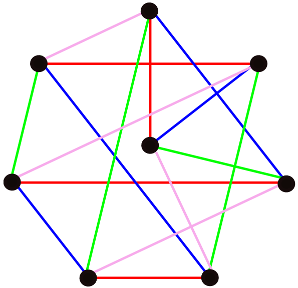

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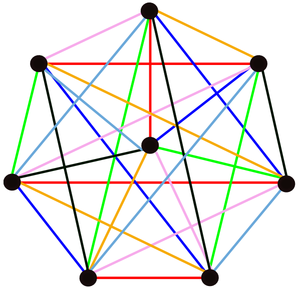
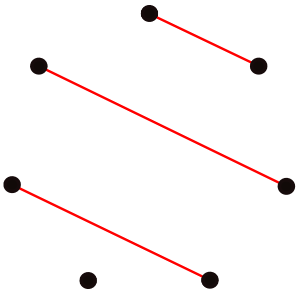


Example of class-2: K_N for odd N



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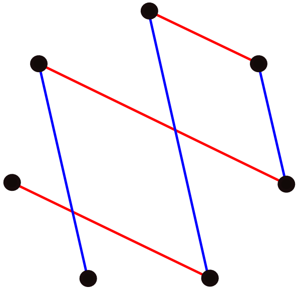

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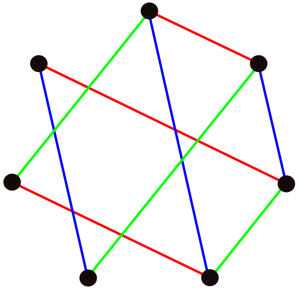

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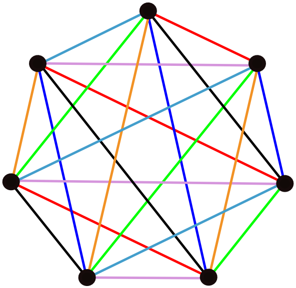
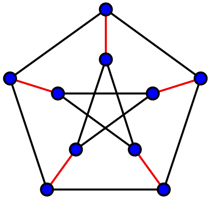


Edge-Coloring and Matchings

- Edge-coloring and partition $\{M_1, ..., M_k\}$ of $E(G)$ into matchings are equivalent.
- d-regular graphs of class 1 have a partition into perfect matchings.
- d-regular graphs of class 2 may have a perfect matching. Example: Petersen graph ($\chi' = 4$, a snark)



Edge-Coloring and Hamiltonian Cycles [1]

- Cubic graphs with a Hamiltonian cycle are 3-edge colorable.
- A uniquely 3-edge colorable cubic graph must have exactly three Hamiltonian cycles.
- The converse is not true: there are cubic graph with three Hamiltonian cycles with $\chi' = 4$.

[1] W.T. Tutte, Hamiltonian circuits, Colloquio Internazional sulle Teorie Combinatorics, Atti dei Convegni Lincei. 17 (1976), Accad. Naz. Lincei, Roma I, 193-199.

Edge-Coloring

Fact

1. *To determine whether an arbitrary graph is of class 1 is NP-complete (even for cubic graphs).*
2. *There are asymptotic results in literature showing that the proportion of random graphs of class 2 is very small.*
3. *Given a graph G of class 2, two ways to modify the graph in order to create a new graph of class 1:*
    3.1 *Add a leaf to each vertex of G*
    3.2 *Make an identical copy of G and add edges connecting the pairs of identical vertices*

Edge-Coloring and multigraphs

Extension of Vizing's theorem for multigraph:

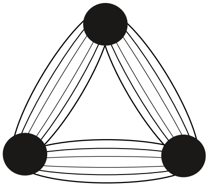
Theorem

(Vizing) For any loopless multigraph G,

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + \mu(G),$$

where $\mu(G)$ is the maximum number of parallel edges joining two vertices (multiplicity of G).

Example: $\chi' = 18$, $\Delta = 12$, $\mu = 6$



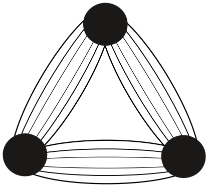
Edge-Coloring and multigraphs

Theorem

(Shannon) For any loopless multigraph G,

$$\Delta(G) \leq \chi'(G) \leq \frac{3}{2}\Delta(G).$$

Example: $\chi' = 18$, $\Delta = 12$



Directed Graphs

Definition

1. A *directed graph* or *digraph* $D(V, A)$ is defined by a vertex set $V(D)$, an arc set $A(D)$, and a function assigning each arc an ordered pair of vertices.
2. A *directed edge* or *arc* (v, v') is an ordered pair of vertices, where v is the *tail* and v' is the *head*.
3. A digraph is a *simple* if each ordered pair is the head and tail of at most one arc.
4. The *underlying graph* $G(D)$ of a digraph D has the same vertex set and is the graph obtained from D by replacing each arc by an edge (opposite arcs are replaced by one edge).
5. The *associated symmetric digraph* $D(G)$ of a graph G is obtained from G by replacing each edge by a pair of opposite arcs.

Directed Graphs - example

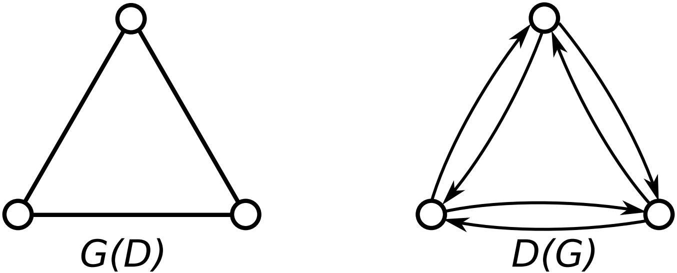


Figure: Underlying simple graph $G(D)$ and the associated symmetric digraph $D(G)$.

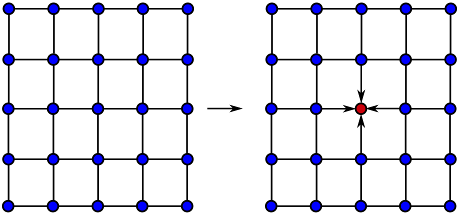
Directed Graphs - more definitions

Definition

1. The *outdegree* $d^+(v)$ is the number of arcs with tail v.
2. The *indegree* $d^-(v)$ is the number of arcs with head v.
3. A *local sink* or simply *sink* is a vertex with outdegree zero.
4. A *local source* or simply *source* is a vertex with indegree zero.
5. A *global sink* is a vertex which is reached by all other vertices.
6. A *global source* is a vertex which reaches all other vertices.

Directed Graphs and spatial search

Marking a vertex using directed graphs:



This technique is used in Szegedy's model ($A_G \to A_{G'}$)

Final comments

- We have reviewed the basics of graph theory
- We have given details about cliques and clique graphs
- We have given details about line graphs
- We have reviewed edge-coloring
- In the next days, we will use graph theory to define:
  - The coined quantum walk model
  - The staggered quantum walk model
  - Spatial search problem
  - Analyze equivalence of QWs

Thank you

Questions?