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Quantum Walks and Graph Coloring

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Thursday

Outline of the Mini-course

- Review on graph coloring
- Coined quantum walks on graphs
- Staggered quantum walks on graphs
- **Spatial search algorithm**
- Equivalence of discrete-time QWs

Today's Outline

- Search on class-1 graphs using the coined model
- Review of the amplitude amplification method
- Search on class-2 graphs using the coined model
- Search on graphs using the staggered model

Spatial search algorithm – Class-1 graphs

Let G be a d-regular Class-1 graph, where $d = \chi'(G)$. A quick review of the definition of the coined model is

Definition

1. The Hilbert space associated with G is $\mathcal{H}_V \otimes \mathcal{H}_C$.

- The evolution operator is $U = S \cdot (I \otimes C)$.
- The flip-flop shift operator is $S|v, c\rangle = |v', c\rangle$, where the color of edge $\{v, v'\}$ is c.
- $I \otimes C$ is the coin operator. The Grover coin is

$$C = 2|u\rangle\langle u| - I_d$$

where

$$|u\rangle = \frac{1}{\sqrt{d}} \sum_{c=1}^{d} |c\rangle.$$

Spatial search algorithm [1]

- The spatial search algorithm in the coined model uses a modified coin operator C' defined as

$$C' = (I_N - |w\rangle\langle w|) \otimes C + |w\rangle\langle w| \otimes (-I_d),$$

  where C is the Grover coin and w is the marked vertex.
- The evolution operator of the spatial search algorithm is $U' = SC'$.
- The shift operator is the same as before: $S|v, c\rangle = |v', c\rangle$, where the color of edge $\{v, v'\}$ is c.
- Initial state is the uniform superposition: $|\psi(0)\rangle = |u\rangle_P \otimes |u\rangle_C$.

[1] N. Shenvi, J. Kempe, B. Whaley. Quantum Random Walk Search Algorithm. Phys. Rev. A 67, 052307, 2003.

Spatial search algorithm

- The state of the walk after t steps is

$$|\psi(t)\rangle = (U')^t\,|\psi(0)\rangle.$$

- The probability of finding w is after t steps is

$$p_w(t) = \sum_{c=1}^{d} \left|\langle w, c|\psi(t)\rangle\right|^2 .$$

- Usually

$$p_w(t) = p_{\max}\sin^2 \epsilon t,$$

where ϵ depends on the spectral gap of U', which can be obtained from the full spectrum of U.

- The running time is $t_{\text{opt}} = \frac{\pi}{2\epsilon}$ and $p_{\max}$ is the success probability.

Spatial search algorithm

- Usually $p_{\max}$ decreases when N increases. (Let $N = |V(G)|$)
- Applying the amplitude amplification method, the success probability $p_{\text{succ}} \longrightarrow 1$ by an overhead of $\frac{1}{\sqrt{p_{\max}}}$ times the running time. That is

$$t_{\text{run}} = \frac{t_{\text{opt}}}{\sqrt{p_{\max}}}.$$

Example: On the 2-dimensional lattice, $p_{\max} = O\left(\frac{1}{\log N}\right)$ and $t_{\text{opt}} = O\left(\sqrt{N \log N}\right)$. Then, the total running time with success probability $\Omega(1)$ is

$$t_{\text{run}} = O\left(\sqrt{N} \log N\right).$$

Alternative formulation of the spatial search: $SC' = UR$

Consider this little algebra, where $C = 2|u\rangle\langle u| - I_d$:

- $SC' = S \cdot \left[(I_N - |w\rangle\langle w|) \otimes C + |w\rangle\langle w| \otimes (-I_d) \right]$
- $\quad = S \cdot \left[I_N \otimes C - |w\rangle\langle w| \otimes (C + I_d) \right]$
- $\quad = S \cdot \left[I_N \otimes C - |w\rangle\langle w| \otimes (2|u\rangle\langle u|) \right]$
- $\quad = U - 2S \cdot (|w\rangle\langle w| \otimes |u\rangle\langle u|)$
- $\quad = U - 2S \cdot (I_N \otimes C) \cdot (|w\rangle\langle w| \otimes |u\rangle\langle u|)$
- $\quad = U \cdot (I - 2|w\rangle\langle w| \otimes |u\rangle\langle u|)$
- $\quad = U\,R$

where

$$R = I - 2|w\rangle\langle w| \otimes |u\rangle\langle u|.$$

R inverts the phase of the marked vertex "$|w\rangle|u\rangle$":

$$R|v\rangle|u\rangle = \begin{cases} -|w\rangle|u\rangle, & \text{if } v = w \\ |v\rangle|u\rangle, & \text{otherwise} \end{cases}$$

where $v \in V$.

Formalism of the spatial search on class-1 graphs

The evolution operator is

$$U' = UR,$$

where U is the evolution operator of the QW on G with no marked vertices and the oracle is

$$R = I - 2\big|w'\big\rangle\big\langle w'\big|,$$

where $|w'\rangle = |w\rangle \otimes |u\rangle$ and $|u\rangle$ is the unit uniform vector on the coin space.

Let's go to class-2 graphs $\rightarrow$

The spatial search on class-2 graphs – arc notation

- The Hilbert space is $\mathcal{H} = \text{span}\{|a\rangle : a \in A(G)\}$.
- We employ formula: $U' = U\,R$
- We easily write U in the arc notation.
- Define the oracle R as

$$R = I - 2\big|w'\big\rangle\big\langle w'\big|$$

where

$$\big|w'\big\rangle = \frac{1}{\sqrt{d^{+}(w)}} \sum_{\substack{a\in A \\ \text{tail}(a)=w}} |a\rangle$$

Hint: $|w\rangle \otimes |u\rangle = \frac{1}{\sqrt{d}} \sum_c |w, c\rangle \longrightarrow |w'\rangle$

The spatial search on class-2 graphs – arc notation

- The initial state is the uniform superposition on $A(G)$

$$|\psi(0)\rangle = \frac{1}{\sqrt{|A|}} \sum_{a \in A} |a\rangle.$$

- The state of the modified QW after t steps is

$$|\psi(t)\rangle = (U')^t |\psi(0)\rangle.$$

- The success probability is

$$p_{\mathrm{succ}} = p_w(t_{\mathrm{opt}}) = \sum_{\substack{a \in A \\ \mathrm{tail}(a)=w}} \left|\left\langle a \middle| \psi(t_{\mathrm{opt}})\right\rangle\right|^2.$$

- In some graph classes we may write

$$p_{\mathrm{succ}} = \left|\left\langle w' \middle| \psi(t_{\mathrm{opt}})\right\rangle\right|^2.$$

Review on amplitude amplification method

Notes:

- If the success probability of one shot of a classical algorithms is p, then the success probability after $\frac{1}{p}$ shots is $\Omega(1)$.
- In the quantum case, the success probability is $\Omega(1)$ after $\frac{1}{\sqrt{p}}$ shots.

Method:

- Let operator A be our algorithm, and $|\psi_{\text{in}}\rangle$ the input. The success probability is $p = |\langle w|A|\psi_{\text{in}}\rangle|^2$.
- Let $|\psi\rangle = A|\psi_{\text{in}}\rangle$.
- Define $U = (2|\psi\rangle\langle\psi| - I)R_w$.
- The new optimal running time is

$$t_{\text{run}} = \left\lfloor \frac{\pi}{4\sqrt{p}} \right\rfloor .$$

- The new success probability is

$$p_{\text{succ}} = 1 - p + O(p^2).$$

Review on amplitude amplification method

Simplest example: Grover's algorithm

- Let $N = 2^n$ and w is the marked element.
- Let $A = H^{\otimes n}$ and $|\psi_{\text{in}}\rangle = |0\rangle^{\otimes n}$.
- The success probability is $p = 1/N$.

Method:

- Define $|\psi\rangle = H^{\otimes n}|0\rangle^{\otimes n}$.
- Let $R_w = I - 2|w\rangle\langle w|$.
- Define $U = (2|\psi\rangle\langle\psi| - I)R_w$.
- The new optimal running time is

$$t_{\text{run}} = \left\lfloor \frac{\pi}{4}\sqrt{N} \right\rfloor .$$

- The new success probability is

$$p_{\text{succ}} = 1 - \frac{1}{N} + O\left(\frac{1}{N^2}\right) .$$

The spatial search in the staggered model

Quick review of the staggered QW on a simple graph G:

- $\mathcal{H} = \text{span}\{|v\rangle : v \in V(G)\}$.
- The first step is the obtaining of a tessellation cover $\{\mathcal{T}_1, ..., \mathcal{T}_k\}$.
- Each tessellation $\mathcal{T}$ is associated with $H = 2\sum_{j=1}^{p} |\alpha_j\rangle\langle\alpha_j| - I$, where

$$|\alpha_j\rangle = \frac{1}{\sqrt{|\alpha_j|}} \sum_{\ell \in \alpha_j} |\ell\rangle,$$

  and α_j's are tiles and p is the number of tiles.
- The evolution operator is

$$U = \mathrm{e}^{\mathrm{i}\theta_k H_k} \ldots \mathrm{e}^{\mathrm{i}\theta_1 H_1},$$

  where θ_j are angles.

The spatial search with the oracle

- Define the oracle

$$R = I - 2|w\rangle\langle w|,$$

where w is the marked vertex.

- The modified evolution operator is

$$U' = U\,R.$$

- The state of the walk after t steps is

$$|\psi(t)\rangle = (U')^t\,|\psi(0)\rangle,$$

where $|\psi(0)\rangle$ is the uniform superposition:

$$|\psi(0)\rangle = \frac{1}{\sqrt{|V|}} \sum_{v \in V} |v\rangle.$$

The spatial search with the oracle

- The probability of finding w is after t steps is

$$p_w(t) = \left|\left\langle w \middle| \psi(t) \right\rangle\right|^2.$$

- Usually

$$p_w(t) = p_{\max} \sin^2 \epsilon t,$$

  where ϵ depends on the spectral gap of U', which can be obtained from the full spectrum of U.

- The running time is $t_{\text{opt}} = \frac{\pi}{2\epsilon}$ and $p_{\max}$ is the success probability.

Method to obtain ϵ and $p_{\max}$ (see review in [1] chap. 9)

- The method is a follow up of the *abstract search algorithm* (Ambainis et.al, Tulsi, others).
- The method depends on 3 hypotheses that **cannot** be checked on arbitrary graphs.
- The validation of the method can only be done in restricted graph classes. E.g.: Johnson graphs.
- There are many successful examples in the literature.
- The method can be extended to multiple marked vertices [2].
- The method can be adapted for the continuous-time quantum walk model [3].

[1] R. Portugal. Quantum Walks Search Algorithms. Springer, Cham, 2nd ed, 2018.
[2] G. Bezerra et al. Quantum walk-based search algorithms with multiple marked vertices. Phys. Rev. A 103, 062202 (2021).
[3] P. Lugão et al. Multimarked Spatial Search by Continuous-Time Quantum Walk. ArXiv:2203.14384, 2022.

Method to obtain ϵ and $p_{\max}$: Notation

- Spectral decomposition of U:

$$U = \sum_k \mathrm{e}^{\mathrm{i}\phi_k} |\psi_k\rangle\langle\psi_k|.$$

  Assume that ϕ_k and $|\psi_k\rangle$ are known.
- Spectral decomposition of U':

$$U' = \sum_{\exp(\mathrm{i}\lambda')\in\sigma(U')} \mathrm{e}^{\mathrm{i}\lambda'} |\lambda'\rangle\langle\lambda'|.$$

  Let λ be the least positive non-null eigenvalues' phase.
  $\mathrm{e}^{\mathrm{i}\lambda}$ is the eigenvalue of U' closest to 1 in the 1st quadrant.
  We will show that $\epsilon = \lambda$.

Method to obtain ϵ and $p_{\max}$: Main equations

- First hypothesis: $\lambda \ll \phi_{\min}$ when $N \gg 1$, where $\phi_{\min}$ is the smallest positive value of ϕ_k.
- Using

$$\langle w | \lambda \rangle = \sum_k \langle w | \psi_k \rangle \langle \psi_k | \lambda \rangle$$

and $\langle \psi_k | U' | \lambda \rangle = \langle \psi_k | UR | \lambda \rangle$, we obtain

$$S_0 - S_1 \lambda - S_2 \lambda^2 = O(\lambda^3)$$

where

$$\begin{aligned} S_0 &= 2 \sum_{\phi_k = 0} \left| \langle w | \psi_k \rangle \right|^2, \\ S_1 &= \sum_{\phi_k \neq 0} \frac{\left| \langle w | \psi_k \rangle \right|^2 \sin \phi_k}{1 - \cos \phi_k}, \\ S_2 &= \sum_{\phi_k \neq 0} \frac{\left| \langle w | \psi_k \rangle \right|^2}{1 - \cos \phi_k}. \end{aligned}$$

Method to obtain ϵ and $p_{\max}$: Main equations

- Second hypothesis: $\exp(-\mathrm{i}\lambda)$ is also an eigenvalue of U'.
- Third hypothesis: $S_1 = 0$.
- Then

$$\lambda = \frac{\sqrt{S_0}}{\sqrt{S_2}}.$$

- By knowing λ, we can calculate the missing terms $\langle w|\lambda\rangle$ and $\langle \psi(0)|\lambda\rangle$.
- Then

$$p(t) \;=\; \frac{\left|\langle w|\psi(0)\rangle\right|^2}{S_0\, S_2}\,\sin^2 \lambda t.$$

- To apply the method to the coined model we have to take

$$|w\rangle \longrightarrow |w\rangle|u\rangle,$$

where $|u\rangle$ is the uniform superposition on the coin space.

Method to obtain ϵ and $p_{\max}$: Examples

- Complete graph
- N-dimensional square, triangular, hexagonal cyclic lattices
- Hypercube
- Bipartite graphs
- Johnson graphs
- Element distinctness algorithm
- many more ...

Final comments

- We have reviewed the spatial search algorithm for
  - Coined model on class-1 graphs
  - Coined model on class-2 graphs
  - Staggered model
- We have seen a method to calculate the time complexity of search algorithms
- Tomorrow the focus will be on the equivalence of discrete-time QWs

Thank you

Questions?