CRM Montreal 2022

Quantum Walks and Graph Coloring

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Wednesday

Outline

- Review on graph coloring
- Coined quantum walks on graphs
- **Staggered quantum walks on graphs**
- Spatial search algorithm
- Equivalence of discrete-time QWs

Today's Outline

- Defining graph tessellation cover
- Defining staggered quantum walks
- Locality
- Characterizing 2-tessellable quantum walks
- Showing that Szegedy's QWs are 2-tessellable QWs
- *k*-tessellable quantum walks

Staggered Quantum Walks on Graphs

Properties of the staggered model:

- It is a discrete-time model.
- The spatial structure is a simple graph.
- The locations of the walker are the vertices:

$$\mathcal{H} = \{|v\rangle : v \in V(G)\}.$$

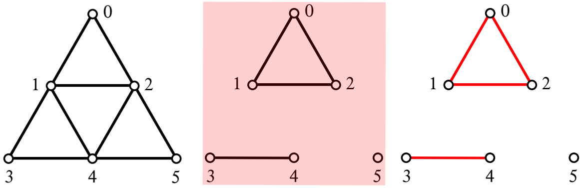
- The model does not have a coin (no inner space).
- The model may employ more than two local operators.
- Each local operator is an extension of the shift operator.

Graph Tessellation Cover

Before defining graph tessellation cover, we need to review:

Partition into cliques:
A *partition* of the vertex set into cliques is a collection of disjoint cliques so that the union of these cliques is the vertex set.

For example: Set $\mathcal{T}_1 = \{\{0, 1, 2\}, \{3, 4\}, \{5\}\}$ (red panel) is a partition of the Hajós graph (Sierpiński Gasket Graph S_2) into cliques. $\mathcal{T}_1$ contains the vertex set but doesn't contain the edge set.



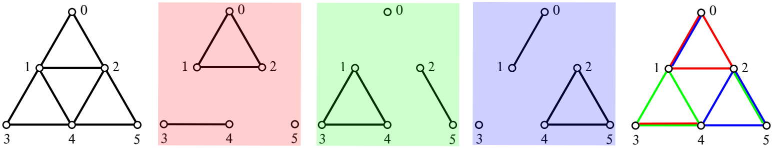
Graph Tessellations

Definition

1. A *graph tessellation* $\mathcal{T}$ is a partition of the vertex set into cliques.
2. An element of the tessellation is called a *tile*.
3. An edge *belongs* to the tessellation $\mathcal{T}$ if and only if its endpoints belong to the same tile.
4. The *size* of a tessellation $\mathcal{T}$ is the number of tiles in $\mathcal{T}$.

For example: $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ are tessellations:

$$\mathcal{T}_1 = \{\{0,1,2\},\{3,4\},\{5\}\} \text{ red}$$
$$\mathcal{T}_2 = \{\{1,3,4\},\{2,5\},\{0\}\} \text{ green}$$
$$\mathcal{T}_3 = \{\{2,4,5\},\{0,1\},\{3\}\} \text{ blue}$$



Graph Tessellation Cover

Definition

Given a graph G with edge set $E(G)$, a **graph tessellation cover** of size k is a set of k tessellations $\mathcal{T}_1, ..., \mathcal{T}_k$, whose union covers the edges, that is,

$$\cup_{i=1}^{k} E(\mathcal{T}_i) = E(G).$$

Example: A tessellation cover of the Hajós graph is $\{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3\}$, where

$$\begin{aligned} \mathcal{T}_1 &= \{\{0,1,2\},\{3,4\},\{5\}\}, \\ \mathcal{T}_2 &= \{\{1,3,4\},\{2,5\},\{0\}\}, \\ \mathcal{T}_3 &= \{\{2,4,5\},\{0,1\},\{3\}\}. \end{aligned}$$

Note that $E(\mathcal{T}_1) \cup E(\mathcal{T}_2) \cup E(\mathcal{T}_3) = E(G)$.

Graph Tessellation Cover

Definition

A graph G is called *k-tessellable* if there is a tessellation cover of size at most k. The size of a smallest tessellation cover of G is called *tessellation cover number* and is denoted by $T(G)$.

Example:

We have provided a tessellation cover of size 3 for the Hajós graph. Then, it is 3-tessellable. An exhaustive inspection shows that it is not possible to find a tessellation cover of size 2 or 1. Then, $T(\text{Hajós}) = 3$.

The Evolution Operator of the Staggered Model

- Let $G(V, E)$ be a connected simple graph so that $|V| = N$ and let $\mathcal{H} = \{|v\rangle : v \in V(G)\}$. Let $\left\{\mathcal{T}_1, ..., \mathcal{T}_k\right\}$ be a tessellation cover of G of size k.

- Suppose that $\mathcal{T}_1 = \{\alpha_j : 1 \leq j \leq p\}$ (α_j is a tile and p is the tessellation size). For each tile, define

$$|\alpha_j\rangle = \frac{1}{\sqrt{|\alpha_j|}} \sum_{\ell \in \alpha_j} |\ell\rangle,$$

where $|\alpha_j|$ is the number of vertices in tile α_j.

- Define the Hermitian and unitary operator associated with $\mathcal{T}_1$ as

$$H_1 = 2 \sum_{j=1}^{p} |\alpha_j\rangle\langle\alpha_j| - I.$$

The Evolution Operator of the Staggered Model

Example: The tiles of tessellation $\mathcal{T}_1 = \{\{0,1,2\},\{3,4\},\{5\}\}$ of the Hajós graph are associated with the vectors

$$|\alpha_1\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + |1\rangle + |2\rangle\right)$$
$$|\alpha_2\rangle = \frac{1}{\sqrt{2}}\left(|3\rangle + |4\rangle\right)$$
$$|\alpha_3\rangle = |5\rangle$$

and tessellation $\mathcal{T}_1$ is associated with the Hermitian operator

$$H_1 = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} & 0 & 0 & 0 \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The Evolution Operator of the Staggered Model

The evolution operator of the staggered model associated with a tessellation cover $\{\mathcal{T}_1, ..., \mathcal{T}_k\}$ is

$$U = \mathrm{e}^{\mathrm{i}\theta_k H_k} \ldots \mathrm{e}^{\mathrm{i}\theta_1 H_1},$$

where θ_j are angles and H_j is associated with tessellation $\mathcal{T}_j$.

Since $H_j^2 = I$,

$$\mathrm{e}^{\mathrm{i}\theta_j H_j} = \cos\theta_j \, I + \mathrm{i} \sin\theta_j \, H_j.$$

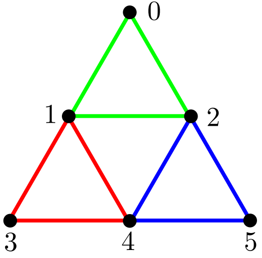
A staggered quantum walk based on a k-tessellable graph is called *k-tessellable quantum walk*.

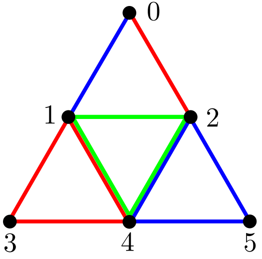
Summarizing the Definition of the SQW

1. Given a graph G, find a tessellation cover $\{\mathcal{T}_1, ..., \mathcal{T}_k\}$. The least k, the better.
2. Associated with $\mathcal{T}_1 = \{\alpha_1, ..., \alpha_p\}$, define
$$H_1 = 2\sum_{j=1}^{p} |\alpha_j\rangle\langle\alpha_j| - I, \text{ where } |\alpha_j\rangle = \frac{1}{\sqrt{|\alpha_j|}} \sum_{\ell\in\alpha_j} |\ell\rangle,$$
and so on for H_2, ..., H_k.
3. The evolution operator is
$$U = \mathrm{e}^{\mathrm{i}\theta_k H_k} \ldots \mathrm{e}^{\mathrm{i}\theta_1 H_1}.$$
4. The state of the walk after t steps is $|\psi(t)\rangle = U^t|\psi(0)\rangle$, where $|\psi(0)\rangle$ is the initial state.
5. The probability of finding the walker on vertex v after t steps is
$$p_v(t) = \left|\langle v|\psi(t)\rangle\right|^2.$$

Alternative Tessellation Covers

Example of an alternative tessellation covers of the Hajós graph:





$\mathcal{T}_1 = \{\{0,1,2\},\{3\},\{4\},\{5\}\}$ $\quad$ $\mathcal{T}'_1 = \{\{1,2,4\},\{0\},\{3\},\{5\}\}$

$\mathcal{T}_2 = \{\{1,3,4\},\{0\},\{2\},\{5\}\}$ $\quad$ $\mathcal{T}'_2 = \{\{1,3,4\},\{0,2\},\{5\}\}$

$\mathcal{T}_3 = \{\{2,4,5\},\{0\},\{1\},\{3\}\}$ $\quad$ $\mathcal{T}'_3 = \{\{2,4,5\},\{0,1\},\{3\}\}$

Note that

$$E(\mathcal{T}_i) \cap E(\mathcal{T}_j) = \emptyset \quad \text{but} \quad E(\mathcal{T}'_1) \cap E(\mathcal{T}'_2) = \{14\} \;\; E(\mathcal{T}'_1) \cap E(\mathcal{T}'_3) = \{24\}$$

Those tessellations covers have different dynamics.

Locality in the Staggered Model

Definition

A linear operator H is *local* with respect to a graph G when $\langle v_2 | H | v_1 \rangle = 0$ if vertices v_1 and v_2 ($v_1 \neq v_2$) are non-adjacent.

Notes:

- H_1 is *local* because

$$\langle v_2 | H_1 | v_1 \rangle = 2 \sum_{j=1}^{p} \langle v_2 | \alpha_j \rangle \langle \alpha_j | v_1 \rangle - \langle v_2 | v_1 \rangle = 0.$$

  If vertices v_1 and v_2 are non-adjacent (a clique contains either v_1 or v_2).

- $\mathrm{e}^{\mathrm{i}\theta H_1}$ is also local because $\mathrm{e}^{\mathrm{i}\theta_j H_1} = \cos\theta_1 \, I + \mathrm{i} \sin\theta_1 \, H_1$.

More on Graph Tessellation Cover

- Recall that $U = e^{i\theta_k H_k} \ldots e^{i\theta_1 H_1}$.
- The number of local operators is the size of the tessellation cover
- Which graphs are 2-tessellable?

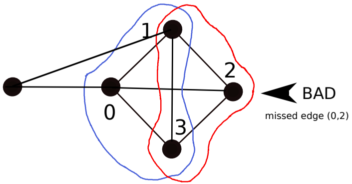
Which graphs are 2-tessellable?

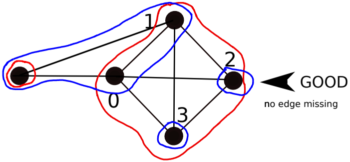
Lemma

Each maximal clique of a 2-tessellable graph $G \neq K_N$ is inside in exactly one tile.

Proof.

It is impossible to partition a maximal clique C into two smaller cliques C_a and C_b such that $C = C_a \cup C_b$. Then, C must be inside one tile of $\mathcal{T}_1$. If tessellation $\mathcal{T}_1$ covers C and $G \neq K_N$, the tessellation $\mathcal{T}_2$ must cover a vertex that does not belong to C. Then $\mathcal{T}_2$ does not cover C. Then C is in exactly one tile. □





Which graphs are 2-tessellable?

Proposition

G is 2-tessellable if and only if its *clique graph* is 2-*colorable*.

Proof.

("if" part) If the clique graph is 2-colorable, it induces a method to color the maximal cliques of *G* with 2 colors.
("only if" part') If *G* is 2 tessellable, the previous lemma shows that the clique graph is 2-colorable. □

[1] R. Portugal. Staggered Quantum Walks on Graphs. Phys. Rev. A 93, 062335 (2016)
[2] M.C. Golumbic et.al. Edge-intersection graphs of boundary-generated paths in a grid. Discrete Applied Mathematics, 236, 214-222 (2018) – See Theorem 3.1 (using equivalence covering – Duchet)

Which graphs are 2-tessellable?

Answer: The set of graphs G whose clique graph is 2-colorable.

Alternative characterization:

Lemma

(Peterson) $K(G)$ is bipartite if and only if G is the line graph of a bipartite multigraph.

Proposition

G is 2-tessellable if and only if G is the line graph of a bipartite multigraph.

Characterization of 2-tessellable QWs

Proposition

1. G is 2-tessellable without an edge in the tessellation intersection **iff** G is the line graph of a bipartite simple graph **iff** the tessellation cover is a 2-colorable Krausz partition.

2. G is 2-tessellable with an edge in the tessellation intersection **iff** G is a line graph of a bipartite multigraph.

Proof.

(Peterson) G is the line graph of a bipartite graph $\Leftrightarrow$ G is diamond-free and $K(G)$ is bipartite
(Peterson) A graph is diamond-free $\Leftrightarrow$ any two maximal cliques intersect in at most one vertex $\Leftrightarrow$ each edge lies in exactly one maximal clique

□

Note: Szegegy's quantum walks are defined on bipartite graphs.
Cannot be extended in a natural way to bipartite multigraphs.
Question: Are Szegedy's QWs instances of 2-tessellable SQWs?

2-tessellable graphs and Szegedy's model [1]

Let G be a simple graph with transition matrix $p_{x,y}$.

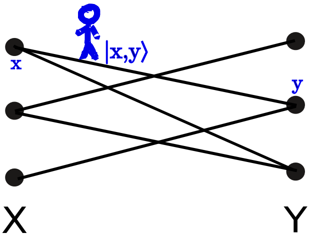
Definition

Szegedy's QW on a bipartite graph $\Gamma(X, Y, E)$ is defined on a Hilbert space $\mathcal{H}^{|X|} \otimes \mathcal{H}^{|Y|}$, whose computational basis is $\{|x, y\rangle : x \in X, y \in Y\}$. The evolution operator is

$$W = R_1 R_0,$$

where

$$R_0 = 2 \sum_{x \in X} |\phi_x\rangle\langle\phi_x| - I$$
$$R_1 = 2 \sum_{y \in Y} |\psi_y\rangle\langle\psi_y| - I$$
$$|\phi_x\rangle = \sum_{y \in Y} \sqrt{p_{xy}}\, |x, y\rangle$$
$$|\psi_y\rangle = \sum_{x \in X} \sqrt{p_{xy}}\, |x, y\rangle$$



[1] M. Szegedy, Quantum speed-up of Markov chain based algorithms, in Proc. of the 45th Symposium on FOCS, 2004.

Szegedy's QWs $\subset$ 2-tessellable QWs

Proposition

($\Rightarrow$) [1] Any instance of Szegedy's QW can be cast into the Staggered model as a 2-tessellable QW with no edge in the intersection of the tessellations.

Proposition

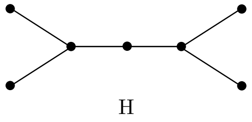
($\Leftarrow$) [1,2] Any instance of a 2-tessellable QWs on line graphs of bipartite graphs with no edge in $E(\mathcal{T}_1) \cap E(\mathcal{T}_2)$ can be cast into Szegedy's QW model.

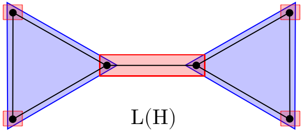
[1] Portugal et al. The staggered quantum walk model, Quantum Inf. Proc. 15, 85 (2016).
[2] R. Portugal. Staggered Quantum Walks on Graphs. Phys. Rev. A 93, 062335 (2016)

Example: 2-tessellable graphs and Szegedy's model

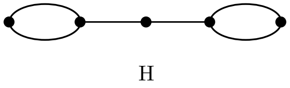
Example of a graph that is a line graph of a bipartite graph:

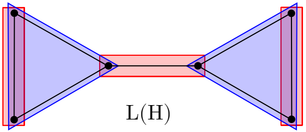
Tessellation cover with no edges in the tessellation intersection





Tessellation cover with edges in the tessellation intersection

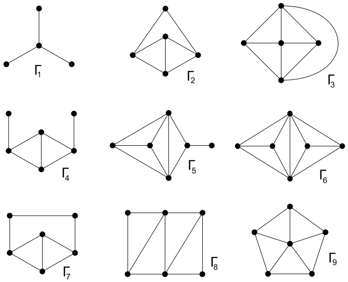




Staggered QWs $\not\subset$ Szegedy's QWs

Proposition

Any staggered QW on G with a forbidden Beineke induced subgraph cannot be cast into Szegedy's QW model.

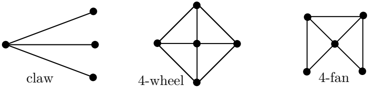


Proposition

1. Only $\Gamma_2, \ldots, \Gamma_6$ are 2-tessellable.
2. If G has induced subgraph Γ_1 or Γ_7 or Γ_8 or Γ_9 then $T(G) \geq 3$.

Staggered QWs $\not\subset$ Szegedy's QWs

Let $\mathcal{F} = \{$claw, 4-wheel, 4-fan, odd cycles$\}$.



Lemma

(Peterson) $K(G)$ is bipartite if and only if G is $\mathcal{F}$-free.

Proposition

G is 2-tessellable if and only if G is $\mathcal{F}$-free.

Proposition

A staggered QW on G can be cast into Szegedy's QW model if and only if G diamond-free and claw-free and has no odd cycle.

What about k-tessellable graphs?

The graph tessellation problem:

Def.: G is *k-tessellable* if there is a tessellation cover of size k.

- The k-TESSELLABILITY problem: Is G is k-tessellable?

Def.: The *tessellation cover number* $T(G)$ is the minimum k such that G is *k-tessellable*.

- Problem: What is the minimum size $T(G)$ of a tessellation cover of G?

- Problem: Given k, is $T(G) \leq k$?

- Problem: What are the bounds for $T(G)$?

Results [1,2]

- $1 \leq T(G) \leq \min\{\chi'(G), \chi(K(G))\}$
- If G is a diamond-free graph with $\chi(K(G)) = \omega(K(G))$, then $T(G) = \chi(K(G))$
- If G is a 3-tessellable diamond-free graph, then $3 \leq \chi(K(G)) \leq 4$
- n-TESSELLABILITY for any fixed $n \geq 4$ is $\mathcal{NP}$-complete
- 4-TESSELLABILITY of chordal $(2, 1)$-graphs is $\mathcal{NP}$-complete
- 3-TESSELLABILITY of planar graphs with $\Delta \leq 6$ is $\mathcal{NP}$-complete
- 3-TESSELLABILITY of diamond-free graphs with diameter five is $\mathcal{NP}$-complete
- 2-TESSELLABILITY can be solved in linear time

[1] A. Abreu, L. Cunha, T. Fernandes, C. de Figueiredo, L. Kowada, F. Marquezino, D. Posner, R. Portugal. The graph tessellation cover number: Chromatic bounds, efficient algorithms and hardness. *Theoretical Computer Science*, Vol 801, (2020) 175-191.
[2] ——. A computational complexity comparative study of graph tessellation problems. *Theoretical Computer Science*, Vol 858, (2021) 81-89.

Final comments

- We have defined graph tessellation cover
- We have defined the staggered quantum walk
- The evolution operator is obtained from a tessellation cover
- Szegedy's model is included in the set of 2-tessellable QWs
- We have discussed k-tessellable quantum walks
- Open problem: How to characterize the class of 3-tessellable graphs?
- Graph theory is essential for quantum walk models
- Tomorrow the focus will be on the spatial search

Thank you

Questions?