CRM Montreal 2022

Quantum Walks and Graph Coloring

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Friday

Outline of the Mini-course

- Review on graph coloring
- Coined quantum walks on graphs
- Staggered quantum walks on graphs
- Spatial search algorithm
- **Equivalence of discrete-time QWs**

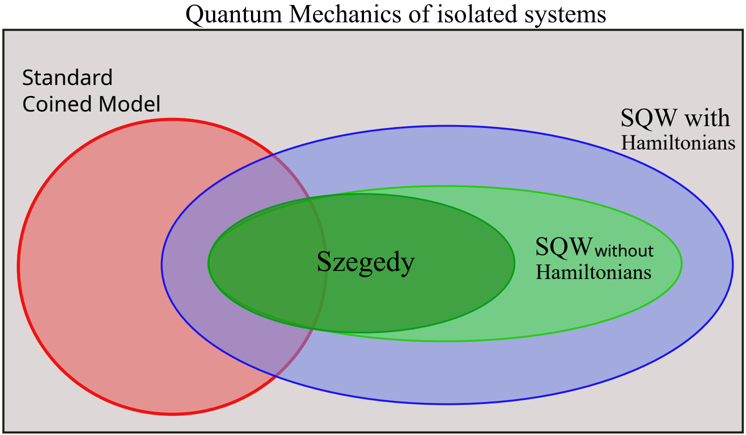
Today's Outline

- What is a quantum walk?
- More on the relation between Szegedy and staggered QWs
- The intersection of coined and staggered QWs
- Search using sinks in Szegedy's model (instead of oracles)
- Search using sinks in the staggered model

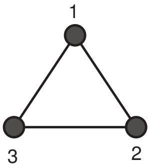
What is a quantum walk?

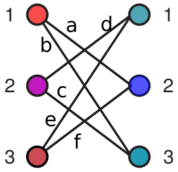
1. It is a dynamical model of a walker on a discrete spatial structure.
2. The dynamics must obey the laws of quantum mechanics and the locality constraints of the spatial structure.
3. Locality and discrete spatial structure are necessary. Without them, QW $\equiv$ QM.
4. Time can be continuous or discrete.
5. The evolution operator must be local if time is continuous.
6. The evolution operator must the product of at least two local operators if time is discrete.
7. In the standard definition of the discrete-time quantum walk, the evolution operator is applied **repeatedly**.
8. In the standard definition of the continuous-time quantum walk, the evolution operator is $\exp(iAt)$, where A is the adjacency matrix or a similar **fixed** matrix of the discrete spatial structure.

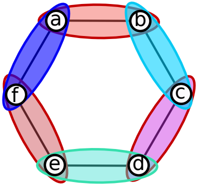
Relation among discrete-time QW models



Szegedy and staggered







Szegedy: $W = R_1 R_0$

$$R_0 = 2\sum_{x\in X} |\phi_x\rangle\langle\phi_x| - I$$
$$R_1 = 2\sum_{y\in Y} |\psi_y\rangle\langle\psi_y| - I$$
$$|\phi_x\rangle = \sum_{y\in Y} \sqrt{p_{xy}}\, |x, y\rangle$$
$$|\psi_y\rangle = \sum_{x\in X} \sqrt{p_{xy}}\, |x, y\rangle$$

Staggered: $U = e^{i\theta H_1} e^{i\theta H_0}$

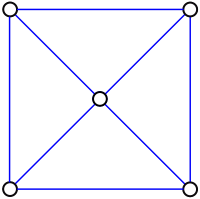
$$H_0 = 2\sum_x |\alpha_x\rangle\langle\alpha_x| - I$$
$$H_1 = 2\sum_x |\beta_x\rangle\langle\beta_x| - I$$
$$|\alpha_x\rangle = \sum_{y\in Y} \sqrt{p_{xy}}\, |xy\rangle$$
$$|\beta_y\rangle = \sum_{x\in X} \sqrt{p_{xy}}\, |xy\rangle$$

Relation between coined and staggered QW models

Definition

Clique-insertion operator [1]: Degree-d vertex is replaced by a d-clique via a blowup process for all d.

Example:



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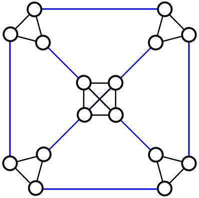


C(G)

[1] Fuji Zhang, et al. Clique-inserted-graphs and spectral dynamics of clique-inserting, J. Math. Anal. Appl., vol 349, 211, 2009.
[2] Chun-Li Kan, et al. Some Chemistry Indices of Clique-Inserted Graph of a Strongly Regular Graph, Complexity, vol. 2021, Article ID 7671212, 2021.

Relation between coined and staggered QW models

Proposition

Let $G(V, E)$ be a graph and let $G'(V', E')$ be the clique-inserted graph $C(G(V, E))$.

1. $|V'| = 2|E|$.
2. E is a perfect matching of G'.
3. G' is 2-tessellable.

Proof.

1. $|V'| = \sum_{v \in V} d(v)$.
2. It follows from the blowup definition.
3. If we remove the edges of G' in E, we obtain a disjoint collection of maximal cliques.

$\square$

Relation between coined and staggered QW models

Proposition

A coined QW on G with flip-flop S and Grover coin is equivalent modulo a global phase to a 2-tessellable QW with $\theta = \pi/2$ on $C(G)$.

Lemma

The Hermitian and unitary operator associated with the complete graph K_N is the N-dimensional Grover coin.

Proof.

The tessellation of K_N has one tile whose unit vector is the uniform superposition $|\psi\rangle$. Then $H_1 = 2|\psi\rangle\langle\psi| - I$, which is the N-dimensional Grover coin. □

Relation between coined and staggered QW models

Proof of the proposition

Idea of the proof:

- The dimension of the Hilbert spaces is $2|E|$ for both models.
- $C(G)$ admits a 2-tessellable QW.
- $C(G)$ has a perfect matching M.
- If we remove the edges of $C(G)$ in M, we obtain a collection of disjoint cliques, which are tiles of tessellation $\mathcal{T}_1$.
- Using the lemma, H_1 is a direct sum of Grover coins.

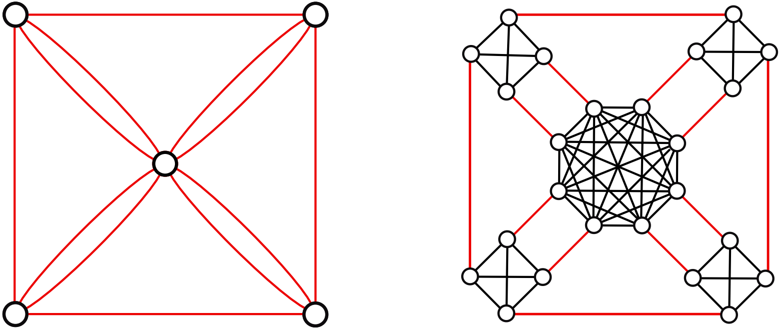
Relation between coined and staggered QW models

continuation of the proof

- The second tessellation $\mathcal{T}_2$ is the perfect matching M.
- Let
$$|\alpha_{vw}\rangle = \frac{|v\rangle + |w\rangle}{\sqrt{2}}$$
be the unit vectors associated with $\{v, w\} \in M$.
- The local Hermitian and unitary operator H_2 associated with tessellation M is
$$H_2 = 2 \sum_{\{v,w\} \in M} |\alpha_{vw}\rangle\langle\alpha_{vw}| - I.$$
- Then,
$$H_2 = \sum_{\{v,w\} \in M} |v\rangle\langle w| + |w\rangle\langle v|,$$
which is a flip-flop shift operator. $\square$

Relation between coined and staggered QW models

The proposition can be extended to multigraphs



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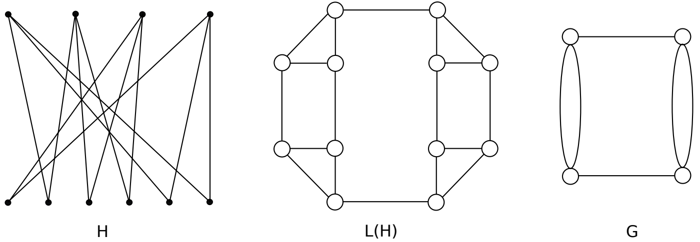
Szegedy, staggered, and coined

Szegedy QWs on bipartite graph with degree-2 vertices in one set are equivalent to coined QWs.

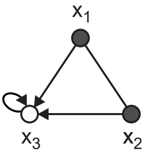


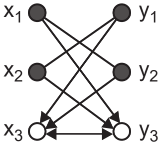
Searching using sinks in Szegedy's model

Let M be the set of marked vertices. Modify the transition matrix as follows:

$$p'_{xy} = \begin{cases} p_{xy}, & x \notin M \\ \delta_{xy}, & x \in M \end{cases}$$

This is equivalent to creating a sink:



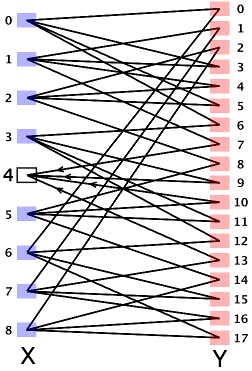


Recal that: $W = R_1 R_0$ where
$R_0 = 2\sum_{x \in X} |\phi_x\rangle\langle\phi_x| - I$
$R_1 = 2\sum_{y \in Y} |\psi_y\rangle\langle\psi_y| - I$
$|\phi_x\rangle = \sum_{y \in Y} \sqrt{p'_{xy}}\, |x, y\rangle$
$|\psi_y\rangle = \sum_{x \in X} \sqrt{p'_{xy}}\, |x, y\rangle$

Searching using sinks in the staggered model

- Vertex 4 is a sink in graph H in Szegedy's QW.

- In the staggered model, remove the tile of the clique associated with the sink.



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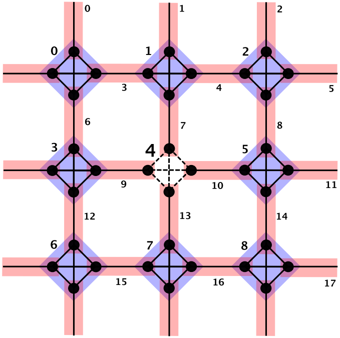

L(H)

Final comments

- We have compared QW models
- There are not equivalent in the standard versions
- Recipes to obtain QWs are useful
- The area of QWs has a long future ahead
- I hope you have enjoyed the quantum walk

The End

Thank you